Where, Oh Waring?
The Classic Problem and its Extensions

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In the 2009-2010 academic year, one of our mathematics majors, Olivia Hightower, became interested in the history of Edward Waring and his famous conjecture about expressing positive integers as the sum of kth powers. Olivia's investigation eventually led to her honors project on Waring's Problem, in which she focused on the history of the conjecture, the eventual proof that all positive integers may be written as the sum of at most nine cubes, and the work of Hardy and Wright in establishing lower bounds in the case of sufficiently large integers. Her research renewed her professor's own interest in Waring, leading to the following article. This paper will sketch brief outlines of Waring's life and the history behind the eventual solution to his problem. In addition, it will present some of the related questions currently being studied, such as expressing sufficiently large integers as sums of powers, sums of powers of primes, and sums of unlike powers.

We begin with a short summary of the biography of Edward Waring. Born in Old Heath, Shropshire in 1736, he excelled in mathematics from an early age. Waring eventually became the sixth Lucasian Chair in Mathematics at Cambridge University, following Isaac Barrow, Sir Isaac Newton, William Whiston, Nicolas Saunderson, and John Colson [9]. In 1770 he published his masterpiece, *Meditationes Algebraicae*, in which he studied symmetric polynomials and cyclotomic equations, stated Wilson's Theorem, and gave the famous generalization of the four squares conjecture [17]. As a member of the Royal Society, Waring was awarded the Copley Medal, but he resigned from the Society in 1795, citing poverty [14]. Waring died just three years later in Pontesbury, Shropshire.

At this point, the interested reader may wish to take the following short quiz on Waring. Answers will be provided at the end of the four questions.

Waring Quiz

1. What was Edward Waring's educational background?
   (a) no formal training—self-taught prodigy
   (b) tried both Oxford and Cambridge but dropped out
   (c) graduated from Oxford with high honors
   (d) graduated from Cambridge with high honors

2. Waring became the Lucasian Professor of Mathematics:
   (a) at age 26, before receiving his M.A.
   (b) at age 30, right after receiving his M.A.
   (c) at age 42, having been passed over once
   (d) at age 58, having been passed over repeatedly
3. Between 1767 and 1770, Waring:
   (a) resigned as Lucasian Professor
   (b) took up the study of medicine
   (c) took up the study of astronomy
   (d) proved the Four Squares Theorem
4. Waring was described as:
   (a) one of the greatest analysts in England
   (b) having an awkward and obscure writing style
   (c) a man of pride and modesty, with pride predominant
   (d) all of the above

Answers (see [9], [14], and [17])

1. (d) Waring attended Magdalene College of Cambridge University, and graduated with high honors.
2. (a) Waring was actually named the Lucasian Chair before officially receiving his master's degree.
3. (b) Waring earned his M. D. but never fully entered into practice as a physician.
4. (d) Each of these descriptions comes from a different contemporary of Waring.

To set the stage for Waring's Problem, we briefly examine the history behind expressing integers as sums of squares, using Burton's text [1] as our main source. One may trace this problem all the way back to Diophantus, although Claude Bachet in 1621 produced the first conjecture that four squares suffice for all positive integers. It should come as no surprise to students of mathematics history that Pierre de Fermat claimed to have a proof of this four-square conjecture yet never shared it. Later, Leonhard Euler would contribute two crucial lemmas, enabling Joseph-Louis Lagrange to complete the proof in 1770.

In that same year, Waring published *Meditationes Algebraicae* and generalized the four-square conjecture to higher powers. He claimed that every positive integer may be written as the sum of at most 9 cubes, at most 19 biquadrates, and so forth. The standard translation of Waring's Problem is: Given $k \geq 2$, there is a least positive integer $g(k)$ such that every positive integer may be written as the sum of $g(k)$ kth powers. In particular, Lagrange proved that $g(2) = 4$, while Waring conjectured that $g(3) = 9$, $g(4) = 19$, etc.; however, it would be many years before mathematicians found a proof of Waring's claims for $k \geq 3$.

Indeed, before 1909, the only known value of $g(k)$ remained $g(2) = 4$. In fact, the only values of k for which the *existence* of $g(k)$ had been established were $k = 2, 3, 4, 5, 6, 7, 8$, and 10. In 1909, David Hilbert produced an impressive breakthrough, proving the existence of $g(k)$ for every $k \geq 2$. Also in 1909, Arthur Wieferich published a proof that $g(3) = 9$; in 1912, Aubrey Kempner filled a gap in Wieferich's proof [13].

Although progress was slow in establishing exact values for $g(k)$, Waring and others quickly found lower bounds. For example, with cubes, 23 provides a "worst-case scenario":

$$23 = 2(2^3) + 7(1^3) \qquad \Longrightarrow \qquad g(3) \geq 9.$$

We note that 23 is not only smaller than 3^3 but also is one less than a multiple of 2^3; similarly, 79 serves as an example for fourth powers:

$$79 = 4(2^4) + 15(1^4) \qquad \Longrightarrow \qquad g(4) \geq 19.$$

In 1772, Johann Euler, the oldest child of Leonhard, generalized this pattern: If $q = \lfloor 3^k/2^k \rfloor$, then

$$g(k) \geq (q - 1) + (2^k - 1) = 2^k + q - 2.$$

According to Delmer and Deshouillers [2], Euler may have believed that equality holds in this lower bound result. Consequently, they label the following claim as

Euler's Conjecture. Given $k \geq 2$, if $q = \lfloor 3^k/2^k \rfloor$, then $g(k) = 2^k + q - 2$.

As previously noted, this conjecture agrees with Lagrange's result $g(2) = 4$ and Waring's claims of $g(3) = 9$ and $g(4) = 19$.

In their survey article [16] on Waring's Theorem, Vaughan and Wooley provide a summary of the history of establishing Euler's Conjecture for various values of k. In 1936, Dickson and Pillai showed that the conjecture holds for $7 \leq k \leq 100$; four years later, Pillai proved it for $k = 6$ as well. In 1957, Mahler made a crucial contribution by showing that the conjecture fails for at most finitely many values of k. Stemmler extended the range to $6 \leq k \leq 200,000$ in 1964, while Chen proved the case $k = 5$ the next year. In 1986, Balasubramanian, Deshouillers, and Dress established the case $k = 4$. Three years later, Kubina and Wunderlich showed that the conjecture holds for $4 \leq k \leq 471,600,000$.

We now shift our attention from expressing every positive integer as a sum of kth powers to expressing every *sufficiently large* integer as such a sum. For $k = 2$, this still requires four squares, as every positive integer congruent to 7 modulo 8 requires four squares as a sum. However, for $k = 3$, the only positive integers to require nine cubes as a sum are 23 and 239. In fact, only finitely many positive integers require eight cubes as a sum. To answer the question of finding the smallest number of cubes needed to express all sufficiently large integers as a sum of cubes (as well as generalizing to kth powers), we define $G(k)$ as follows.

Definition. Given $k \geq 2$, let $G(k)$ be the least positive integer such that every *sufficiently large* integer may be written as the sum of $G(k)$ kth powers.

We note that $G(k) \leq g(k)$, with equality only for $k = 2$. In addition, as Vaughan and Wooley [16] observe, only two values of $G(k)$ are known, namely $G(2) = 4$ (Lagrange) and $G(4) = 16$

(Davenport). For cubes, Linnik showed in 1942 that $G(3) \leq 7$, while a modulo 9 argument establishes $G(3) \geq 4$. The actual value of $G(3)$ remains one of the most famous open questions in number theory. In 1851, Jacobi [1] conjectured that $G(3) \leq 5$, while the current consensus is that four cubes suffice; in fact, 7,373,170,279,850 may be the largest integer that requires more than four cubes [3].

Hardy and Wright [7] established the following lower bounds for $G(k)$:
(a) If $\theta \geq 2$, then $G(2^\theta) \geq 2^{\theta+2}$.
(b) If $\theta \geq 0$ and p is an odd prime, then

$$G(p^\theta(p-1)) \geq p^{\theta+1} \qquad \text{and} \qquad G(\frac{1}{2}p^\theta(p-1)) \geq \frac{1}{2}(p^{\theta+1}-1).$$

(c) For any $k \geq 2$, $G(k) \geq k+1$.

In 1922, Hardy and Littlewood proved that $G(k) \leq (k-2)2^{k-1}+5$, while the current best upper bound for $G(k)$ for sufficiently large k is due to Wooley (1995):

$$G(k) \leq k\left(\log k + \log\log k + 2 + C \cdot \frac{\log\log k}{\log k}\right),$$

where C is some positive constant [16].

The following lower bounds come from Hardy and Wright's results [7] and are conjectured to be the actual values of $G(k)$; the upper bounds are due to Vaughan and Wooley [16]:

$$6 \leq G(5) \leq 17;\;\; 9 \leq G(6) \leq 24;\;\; 8 \leq G(7) \leq 33;\;\; 32 \leq G(8) \leq 42;\;\; 13 \leq G(9) \leq 50.$$

Indeed, much work remains to be done in calculating $G(k)$, even for small values of k.

In order to discuss another variant of Waring's Problem, we first state the following definitions. Given a subset A of the set of positive integers, let $A(n)$ be the number of elements in A which are less than or equal to n. Then the *natural density* of A is

$$\delta(A) = \lim_{n\to\infty} \frac{A(n)}{n}.$$

In particular, if A is the set of positive integers having a property P, then *almost every* positive integer has property P if and only if $\delta(A) = 1$. This motivates the next step, as found in [16].

Definition. Given $k \geq 2$, let $G_1(k)$ be the least positive integer such that *almost every* positive integer may be written as the sum of $G_1(k)$ kth powers.

We note that $G_1(k) \leq G(k) \leq g(k)$. Again following Vaughan and Wooley [16], we observe that only six values of $G_1(k)$ are known:

$G_1(2) = 4$ (Lagrange); $G_1(3) = 4$ (Davenport); $G_1(4) = 15$ (Hardy and Littlewood);
$G_1(8) = 32$ (Vaughan); $G_1(16) = 64$ (Wooley); $G_1(32) = 128$ (Wooley).

For the next version of Waring's Problem, we pause to note two connections between Waring and Christian Goldbach. In 1742, Goldbach conjectured that every even integer greater than two may be written as the sum of two primes, and every odd integer greater than one is either prime or the sum of three primes. Waring became the first to publish this conjecture [17]. Furthermore, Waring's problem and Goldbach's conjecture may be combined into the "Waring-Goldbach problem" [16] as follows: Given $k \geq 1$, let $H(k)$ be the least integer s such that

$$p_1^k + p_2^k + \cdots + p_s^k = n$$

has a solution in primes p_i for all sufficiently large integers n satisfying certain congruence conditions.

In 1937, Vinogradov established his famous result that all sufficiently large odd integers may be written as the sum of at most three primes; equivalently, $H(1) \leq 3$. The next year, Hua [8] proved that $H(k)$ exists for every k, with

$$H(k) \leq k^2(4 \log k + 2 \log \log k + 5)$$

for $k > 10$. Hua also showed that $H(k) \leq 2^k + 1$ for every k; in particular, $H(2) \leq 5$ and $H(3) \leq 9$ remain the best known results in those two cases [11]. In 1987, Thanigasalam established $H(6) \leq 33$ and $H(8) \leq 63$, while in 2001, Kawada and Wooley proved that $H(4) \leq 14$ and $H(5) \leq 21$ (see [16] and [11]). A recent result due to Kumchev [11] gave the current best known upper bound for $H(7)$ as 46, an improvement over the previous "record" of 47.

Many more variations on Waring's Problem exist, including the use of mixed powers. In 1949, Freiman conjectured the following generalization:
Let $\{n_i\}$ be a sequence of integers with $2 \leq n_1 \leq n_2 \leq \ldots$. For any n_j, there exists an integer r such that all sufficiently large integers N are representable in the form

$$N = x_1^{n_j} + x_2^{n_{j+1}} + \cdots + x_r^{n_{j+r-1}}$$

for positive integers x_i if and only if $\sum_{i=1}^{\infty} n_i^{-1} = \infty$.

In 1960, Scourfield proved this result, and in 1966, Thanigasalam established a similar result for prime powers [15]. As noted in Ford's paper [4], in 1951, Roth considered the special case of mixed powers using successive powers. He showed that there is a number s such that all sufficiently large integers N may be written as a sum of s successive powers, starting with a square; that is,

$$N = \sum_{i=1}^{s} x_i^{i+1} = x_1^2 + x_2^3 + \cdots + x_s^{s+1}.$$

Roth not only showed that $s = 50$ works but also established that for *almost all* integers, s can be taken to be 3. Roth's upper bound of 50 for s has subsequently been improved [5], as follows:

Thanigasalam, 1968: $s \leq 35$; Vaughan, 1971: $s \leq 26$;
Thanigasalam, 1984: $s \leq 20$; Brüdern, 1988: $s \leq 17$; Ford, 1996: $s \leq 14$.

It is conjectured that $s = 3$ should hold in the sufficiently large case, as it does in the almost every case [4].

We mention just three more of the variations of Waring's Problem currently being researched.

1. J. Liu, M. Liu, and Zhan [12] have studied expressing integers as sums of squares of primes and powers of 2: $N = p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2^{v_1} + \cdots + 2^{v_k}$.

2. Ford [6] has researched the question of polynomial summands: Given $f(x) \in \mathbb{Z}[x]$ with no fixed integer divisor $d \geq 2$, write all sufficiently large integers N in the form $N = f(x_1) + f(x_2) + \cdots + f(x_s)$, where each x_i is a positive integer.

3. Finally, Kononen [10] has obtained results for solutions over finite fields: Given the finite field $\mathbb{F}_q$, find the minimum s such that we may write every element in $\mathbb{F}_q$ in the form $x_1^k + x_2^k + \cdots + x_s^k$ with $x_i \in \mathbb{F}_q$.

In conclusion, given the remarkable number of generalizations arising from Waring's Problem, we venture the conjecture that Waring himself would have been quite pleased to see so many results extending from where it all began, over 240 years ago:

"Omnis integer numerus vel est cubus, vel e duobus, tribus, 4, 5, 6, 7, 8, vel novem cubis compositus, est etiam quadrato-quadratus vel e duobus, tribus, &c. usque ad novemdecim compositus, & sic deinceps."

– Waring (1770)

"Every integer is a cube or the sum of two, three, ..., nine cubes; every integer is also the square of a square, or the sum of up to nineteen such; and so forth."

– Waring (1991 translation by Weeks [17])

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